Estimating Structural Models of Corporate Bond Prices *

Max Bruche

CEMFI[†]

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Abstract

One of the strengths of structural models (or firm-value based models) of credit (e.g. Merton, 1974) as opposed to reduced-form models (e.g. Jarrow and Turnbull, 1995) is that they directly link the price of equity to default probabilities, and hence to the price of corporate bonds (and credit derivatives). Yet when these models are estimated, the existence of data other than equity prices is typically ignored. This paper describes how all available price data (equity prices, bond prices, possibly credit derivative prices) can be used in estimation, and illustrates that using bond price data in addition to equity price data changes estimates significantly. In this context, the issue of possibly noisy data and/or model error is also discussed.

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[†]Centro de Estudios Monetarios y Financieros, C/ Casado del Alisal 5, 28014 Madrid
Email:`bruche@cemfi.es`

1 Introduction

In 1974, Merton wrote a seminal paper that elaborated some of the implications of the paper by Black and Scholes (1973) for the pricing of corporate debt. Many extensions of this model followed. This family of models is sometimes referred to as the family of structural models of corporate debt, and views corporate debt and equity as options on the fundamental value or asset value of the firm[1].

Structural models have been successfully applied in risk management (e.g. by KMV, see Crosbie and Bohn, 2003). They have, however, been less popular in the pricing of corporate bonds and credit derivatives, because there is evidence that simpler versions underpredict spreads (Jones, Mason, and Rosenfeld, 1984), and more importantly, that even the more elaborate structural models that predict the correct level of spreads on average are very imprecise (Eom, Helwege, and Huang, 2004). The literature has judged structural models by their ability to match bond prices, and it appears that the models do very badly along this dimension.

It is somewhat surprising that given that the models are judged on their ability to match bond price data, this data is typically not used in estimation. Given the strong link predicted by structural models between equity prices and bond prices the question should really be whether structural models can fit the data, given that they are estimated on *all* available data, such as equity prices and bond prices.[2] A second question that immediately follows is whether including bond price information makes a difference at all, and whether or not it is safe to ignore bond price data when estimating,

[1]Extensions of the original model relate to e.g. sub-ordination arrangements, indenture provisions and default before maturity (Black and Cox, 1976), coupon bonds (Geske, 1977), stochastic interest rates (Longstaff and Schwartz, 1995) or an optimally chosen capital structure (Leland, 1994), to name but a few.

[2]This could be extended to spreads of credit default swaps, since structural models predict these, too.

as has been done in the literature so far. It is these questions that this paper adresses. As a byproduct, it is discussed to what extent it is necessary or useful to assume observation errors in bond or equity prices.

The paper shows that when estimating on both equity and bond prices, the models typically have bond pricing errors that are an order of magnitude larger than the corresponding equity pricing errors. This reiterates the previous results in this literature that indicate that there is ample scope for improvement in the structural framework, at least in terms of bond pricing performance. However, bond pricing errors are reduced, when both types of prices are used in the estimation, at the cost of higher estimates of asset value volatilities.

Comparing the estimated asset value volatilities in the case where only equity prices are used in estimation versus the case where both equity and bond prices are used with a Hausman test, it is shown that including bond price information in addition to equity prices in the estimation leads to significantly different estimates. Including bond price data in the estimation also seems to reduce the equity pricing errors of the models, sometimes quite dramatically. This in turn means that whatever the merit of structural models, if they are estimated, both types of prices should be used. This would be especially important in applications which rely on estimates of the asset value volatility in e.g. the calculation of default probabilities.

Lastly, allowing for the models to price equity imperfectly in the estimation seems to have a small effect on the estimated parameters.

The rest of this paper is organised as follows: First, the fundamental problem of estimation is discussed and the common implementation approaches are examined (section 2). A general setup of an estimation is then presented (section 3), and various different approaches to estimation discussed (section 4). Using some theoretical models (section

5), a selection of different methods of estimation are then applied to bond and equity price data (section 7). Section 8 concludes.

2 Traditional Approaches to the Fundamental Problem of Estimating Structural Models

Only in very few special cases is the estimation of structural models straightforward. Gemmill (2002), for instance, picks data on British closed-end funds that issue zero coupon bonds. For this data set, asset values of the funds are readily available (indeed, they are published daily), and the entire debt of the entity consists of one zero coupon bond. This situation exactly matches the assumptions of the Merton model, and direct computation of theoretically predicted bond prices is very simple and raises few problems.

For normal companies, however, asset values are observed very infrequently. Balance sheet information on asset values exists, but it is available at most at a quarterly frequency, and often only at an annual frequency. It has been suggested that the asset value can be calculated by adding the value of equity and the value of debt (this should work for all structural models that produce a capital-structure irrelevance result). If both the market value of equity and the market value of aggregate debt were available, this would be a feasible strategy. However, while market values of equity are generally available for large firms, only a small fraction of debt is typically traded. This has led many authors to form an ad-hoc estimate of the asset value by adding the market value of equity and the *book value* of debt (Jones, Mason, and Rosenfeld, 1984; Ogden, 1987; Anderson and Sundaresan, 2000; Eom, Helwege, and Huang, 2004). Of course, there is no reason to suspect that this ad-hoc calculation yields a particularly accurate estimate of the asset value, especially not for the purpose of pricing market instruments.

In general, the answer to this problem has been to use asset values implied by actual market prices, in particular, equity-implied asset value. For example, Ronn and Verma (1986) suggested solving a set of two equations relating the observed price of equity and estimated (i.e. usually historical) equity volatility to asset value and asset value volatility. The equations used for this are the option-pricing equation describing the value of equity as an option on the underlying asset value (F_E), and the equation describing the relationship between equity volatility and asset value volatility derived from the equity pricing equation via Itō's lemma.

$$E = F_E(V, \sigma_V) \quad (1)$$

$$\sigma_E = \sigma_V \frac{V}{E} \frac{\partial F_E}{\partial V}. \quad (2)$$

This approach is often the only one discussed in major textbooks (Hull, 2003).

Note that when the volatility of equity returns is calculated from historical data, this is typically done assuming that the volatility is constant. Of course, this contradicts [2]. Since the equity volatility changes as the ratio of the value of equity to the value of assets changes, and as the derivative of the equity pricing function changes, this problem will be especially apparent when the asset value (or the leverage) of the firm changes a lot over the estimation period. This approach cannot utilize the dynamic implications of structural models in estimation.

The problems that the method of Ronn and Verma (1986) has with changing leverage has been demonstrated in a Monte Carlo experiment by Ericsson and Reneby (2005), who propose to use a method suggested by Duan (1994) instead: If the asset value (V) follows a geometric Brownian motion, the density of asset values will be Gaussian. Since the model produces a one-to-one mapping from asset values to equity prices E, the density (and hence likelihood) of equity prices can be obtained by a simple change of variables, i.e. the density of asset values needs to be multiplied by the determinant of the Jacobian of the inverse of the pricing function.

$$f(E) = f(V)\frac{dV}{dE} \tag{3}$$

where $f(\cdot)$ denotes a density. The log likelihood can be evaluated by first calculating the equity-implied asset value simply by inverting the relation $E = g(V)$ to obtain $V_E = g^{-1}(E)$, and then calculating

$$\log L(E) = \log f(V_E) - \log\left(\left.\frac{dg(V)}{dV}\right|_{V=V_E}\right) \tag{4}$$

Typically, after calculating asset values in one of the ways described, these are then used to calculate model-implied bond spreads, which can be compared to actual bond spreads.

The important drawback of this method is that it is not possible to use e.g. bond spreads (or other data that might contain information about asset values). In thinking about how to combine the information about asset values contained in several observed prices, it will be useful to view structural models as a state space system.

3 A State Space Representation

Structural models can be viewed as a state space system, which consists of an observation equation which links a vector of observed variables, e.g. logs of bond and equity prices (y_t), to an unobserved state (the asset value of the firm V) via some pricing functions $g(\cdot)$, and a state equation, which describes the dynamics of the state:

$$dV_t = \mu(V_t, t)dt + \sigma(V_t, t)dW \tag{5}$$

$$y_t = g_t(V_t)$$

Note that here, $g(\cdot)$ now denotes a vector of pricing functions.

If the asset value V follows a geometric Brownian motion (for example), we could define $\alpha_t = \int_0^t d\log V$ and the function $Z_t(\alpha) = g(V_t)$, and produce the discrete-time version

$$\alpha_t = d + \alpha_{t-1} + \eta_t \tag{6}$$
$$y_t = Z_t(\alpha_t).$$

The parallels to the estimation of risk-free term structure models are immediately obvious: instead of factors we have the asset value, and instead of observed yields we have observed log prices. To some extent it is surprising that the literature that discusses estimation of risk-free term structure models has not had much influence on the estimation of structural models.[3]

There are some differences vis-a-vis the estimation of risk-free term structure models. For example, in the case of risk-free term structure models, the observation equation is typically much simpler since yields are e.g. affine, or quadratic in the factors. For a structural model, $Z(\cdot)$ would typically be a vector of complicated non-linear functions. This implies that it is unlikely that Quasi Maximum Likelihood Estimation (e.g. with the Kalman filter on a linearized version of the model) would be as successful as in the case of risk-free terms structure models. There are also factors that make things simpler in the case of structural modes, however. Typically, the transition density of the state α is known. Of course, this makes estimation easier than in those risk-free term structure models in which the transition density of the state is not known in closed form.

If the number of observed variables is larger than the number of states, and the

[3]See e.g. Piazzesi (2005), section 6, for a survey of the methodologies used in the context of affine term structure models.

model is well-specified, the variance-covariance matrix of observed variables is singular. Another way of saying this is that for a given set of parameters and a standard structural model, an asset value can be chosen such that at most one observed market price can be matched perfectly.[4] In order to break the stochastic singularity, some form of measurement errors needs to be introduced:

$$\alpha_t = d + \alpha_{t-1} + \eta_t \tag{7}$$
$$y_t = Z_t\left(\alpha_t\right) + \varepsilon_t.$$

These errors can be interpreted as errors in the measurement of the variable in question. They can also be interpreted as model errors: In practice, there are things that are beyond the scope of the model and prices contain market microstructure effects, and reflect liquidity issues, taxes, agency problems etc., which are not included in typical structural models. The issues with the exact measurement of prices are probably outweighed by factors that the models ignore, nevertheless in keeping with standard terminology, these errors will be referred to as measurement errors below.

A simple assumption is that the measurement errors in log prices are normal, independent of each other, and independent of innovations in α, which (together with the assumption of geometric Brownian motion) would produce

$$\eta_t \sim NID\left\{0, \sigma_\eta\right\}, \tag{8}$$
$$\varepsilon_t \sim NID\left\{0, \Sigma_\varepsilon\right\}$$

where Σ_ε is diagonal (if there is only one state, at most one diagonal element of Σ_ε can be zero). This assumption will be used below. Alternative distributional assumptions

[4]This of course assumes that the mapping from asset values into prices be injective.

would be possible, although care has to be taken that the model is identified.[5] Estimating this system is very closely related to obtaining an estimate of the asset value on the basis of the vector of observed prices y_t, i.e. filtering.

4 Estimation Techniques

There are various ways in which the state space system can be estimated. One class of methods which will not be discussed in detail here is based on matching moments. These methods have been employed extensively in the risk-free terms structure literature.[6] An advantage of moment-based methods in general is that models can be evaluated directly on their ability to match moments of interest in the data. However, if the distributional assumptions made in the models are correct, likelihood-based techniques implicitly choose moment conditions which guarantee asymptotic efficiency.[7] This paper explores likelihood-based methods, which have obtained more traction in the recent literature; this will facilitate comparisons.

[5]It would be possible to put different and more elaborate structure on the measurement errors or components that are outside the structural model, in particular if one had a view on what causes the divergence between model prices and observed prices. For example, liquidity risk has been suggested as an important component in the pricing of corporate bonds (Longstaff, Mithal, and Neis, 2005), and an ad-hoc specification could include proxy variables for liquidity in the specification of the measurement or model error.

[6]For example, GMM can be applied in situations where moments are known in closed form, e.g. moments of zero-coupon yields in affine models. For non-affine models, or situations when the moments of interest are moments of nonlinear functions of zero-coupon yields (e.g. in the coupon bonds and swaps), the Simulated Method of Moments has been employed (Lee and Ingram, 1991; Duffie and Singleton, 1993).

[7]Asymptotic efficiency can also be achieved with moment based methods in the context of the Efficient Method of Moments (EMM) (Gallant and Tauchen, 1996). It has been argued, however, that the dependence of the EMM estimator on a non-parametric estimate of the unknown likelihood function can lead to bad small-sample behaviour, at least for risk-free term structure models, and that even approximate filtering/ likelihood-based techniques produce better results (Duffee and Stanton, 2004).

The Duan (1994) method can be interpreted as estimating the system

$$\begin{aligned} \alpha_t &= d + \alpha_{t-1} + \eta_t \\ \log(E) &= Z_t^E(\alpha_t), \end{aligned} \tag{9}$$

where E is the equity price and $Z_t^E(\cdot)$ is the function that maps the log asset value to the log equity price. Note that in this formulation, there are no measurement errors (or their variance is set to zero).

One could think of generalizing this approach by extending the vector of observed variables, but maintaining the assumption that equity prices are observed without error, i.e. estimate the system

$$\begin{aligned} \alpha_t &= d + \alpha_{t-1} + \eta_t \\ y_t &= Z_t(\alpha_t) + \varepsilon_t, \end{aligned} \tag{10}$$

but with the restriction that σ_E, the element of Σ_ε corresponding to the equity price, is zero. Letting $\varepsilon_{\setminus E}$ denote the vector of differences between observed variables (excluding the equity price) and model-implied variables (excluding the equity price), calculated on the basis of the equity-implied log asset value, it is easy to see that the log-likelihood function can be written as

$$\log L(E) = \log f(V_E) - \log\left(\left.\frac{dg(V)}{dV}\right|_{V=V_E}\right) + \log f\left(\varepsilon_{\setminus E}\right) \tag{11}$$

This, in fact, is the method first proposed by Chen and Scott (1993).

Another way to generalize the Duan (1994) approach would be to allow for errors in equity prices, but without including more information in the vector of observed variables, such that the system becomes

$$\begin{aligned} \alpha_t &= d + \alpha_{t-1} + \eta_t \\ \log(E) &= Z_t^E(\alpha_t) + \varepsilon_E. \end{aligned} \tag{12}$$

Alternatively, we could just estimate the full system

$$\alpha_t = d + \alpha_{t-1} + \eta_t \tag{13}$$
$$y_t = Z_t(\alpha_t) + \varepsilon.$$

To obtain the marginal density of the observed variables y and hence the likelihood function in the last to cases, we essentially need to integrate out the unobserved variable α:

$$L(\psi) = p(y;\psi) = \int p(\alpha, y; \psi) d\alpha. \tag{14}$$

As an alternative, if the filtered density $p(\alpha|y;\psi)$ were available, it would be possible to work out the density $p(y;\psi)$ and hence the likelihood from the joint density via Bayes' theorem. In essence, there are several alternatives for calculating the true likelihood numerically, including e.g. particle filters, or importance sampling techniques, which produce the same likelihood up to some chosen numerical precision.

This paper uses an importance sampling method described by Durbin and Koopman (1997).[8] With some trivial algebra, it can be shown that given that one has a tractable approximate model with density $g(\cdot)$, the likelihood can be written as

$$L(\psi) = g(y;\psi) \int \frac{p(\alpha, y;\psi)}{g(\alpha, y;\psi)} g(\alpha|y;\psi) d\alpha \tag{15}$$
$$= g(y;\psi) E_{g(\alpha|y;\psi)} \left[w(\alpha, y;\psi) \right], \tag{16}$$

where

$$w(\alpha, y;\psi) = \frac{p(\alpha, y;\psi)}{g(\alpha, y;\psi)}. \tag{17}$$

We can view $g(\alpha|y;\psi)$ as an importance sampler that is used to integrate out the nuisance variable α (cf. e.g. Ripley, 1987). The closer $g(\alpha|y;\psi)$ is to the true $p(\alpha|y;\psi)$,

[8] This is similar to the Efficient Importance Sampling technique applied by Collin-Dufresne, Goldstein, and Jones (2004).

the more efficient this importance sampling will be. We can also interpret the expression as saying that given an approximation, the true density or likelihood can be expressed as the approximate density times a correction factor, where this correction factor is related to the distance between the truth p and the approximation g, and can be calculated numerically. Given an approximate model and its likelihood L_g, we can calculate

$$\hat{L}(\psi) = L_g(\psi)\bar{w}(\psi), \text{ where } \bar{w}(\psi) = \frac{1}{M}\sum_{i=1}^{M} w_i(\psi), \text{ and } w_i(\psi) = \frac{p\left(\alpha^i, y; \psi\right)}{g\left(\alpha^i, y; \psi\right)}, \tag{18}$$

and α^i is drawn from the importance density.[9,10] The accuracy of this numerical approximation of the true likelihood only depends on M, the size of the Monte Carlo simulation.

In practice, a linear Gaussian approximation is chosen such that the density $g(y;\psi)$ can be calculated via the Kalman filter. Durbin and Koopman (2001) describe several methods for deriving an appropriate approximate model. The basic idea of the appropriate method for this case is to iteratively linearize the observation and state equations. We start with a guess of α, linearize the model around this guess, calculate the implied y, and then smooth to obtain our next guess of α, and iterate this until convergence. The linear model in the last step then is the one that has a smoothed density $g(\alpha|y;\psi)$ with the same mode as the corresponding density of the actual model.[11] This ensures that the importance density is similar to the target density, and that importance sam-

[9]There are several ways to do this. The very simple and efficient scheme proposed by Durbin and Koopman (2002) was used in this paper.

[10]In practice, the log transformation of the likelihood is used. This introduces a bias for which a modification has been suggested that corrects for terms up to order $O\left(M^{-3/2}\right)$ (cf. Shephard and Pitt, 1997; Durbin and Koopman, 1997):

$$\log \hat{L}(\psi) = \log L_g(\psi) + \log \bar{w} + \frac{s_w^2}{2M\bar{w}^2}, \tag{19}$$

with $s_w^2 = (M-1)^{-1}\sum_{i=1}^{M}(w_i - \bar{w})^2$.

[11]For a proof, consult Durbin and Koopman (2001) and references cited therein.

pling is efficient. In most structural models, the state equation is already linear and Gaussian if the log asset value is chosen as the state, which simplifies the procedure.

This approximation method does not work well for models with a default boundary, or first passage models, however (e.g. Black and Cox, 1976; Leland, 1994). These models posit that a company defaults as soon as its asset value falls below a default boundary. In practice, we do not observe the log price vector y for firms that are in default. Hence in situations where we calculate the density $p(\alpha|y)$, we implicitly condition on no default, which means that the density which we are trying to approximate with the importance sampler is truncated at the default barrier. The algorithm described above can break down when an iteration reaches the region below the default barrier. To get around this problem, the algorithm was slightly modified: Starting with a large guess for α (ensuring that the starting value is above the mode, and above the truncation point), we iterate until convergence. If convergence is achieved, we will have found an approximate non-truncated Gaussian density with the mode of the true (truncated) density $p(\alpha|y)$. If the truncation point is hit before convergence, the last approximating model in the iteration is chosen, such that the approximating model has the truncation point as its mode.[12]

To summarize, the procedure for evaluating the likelihood of the full state space system with all data and no restrictions on measurement errors is as follows: 1. Calculate a tractable approximation of the model, 2. calculate the likelihood of the approximation, 3. obtain a correction factor by simulating α from the importance density, to numerically calculate the expectation term, 4. combine both to obtain the likelihood.

We are now in a position to compare likelihood-based estimation for different type

[12]This is a somewhat arbitrary choice of importance density, which implies that convergence might be slow. In practice, this means that the number of simulations necessary for a given degree of accuracy rises.

of datasets and assumptions on the measurement errors. In the following, these methods will be compared: I. estimation using equity and bond price data, allowing for measurement errors in equity (using Durbin and Koopman, 1997), II. estimation using equity and bond price data, not allowing for measurement errors in equity ($\sigma_E = 0$) (using Chen and Scott, 1993), III. estimation using only equity data, allowing for measurement errors in equity (using Durbin and Koopman, 1997), and IV.estimation using only equity price data, and not allowing for measurement errors in equity ($\sigma_E = 0$) (using Duan, 1994), as well as V. the method of Ronn and Verma (1986), using only equity data.

Since the method of Durbin and Koopman (1997) has not been applied in the context of structural models, Appendix A presents a Monte Carlo experiment that illustrates that the method as described here works well for this type of application, and that the method of Duan (1994), as well as the method of Ronn and Verma (1986) can have problems in this context, especially in the presence of measurement errors.

5 Theoretical models

Before applying the different methods, a structural model has to be chosen. Here, three models were considered: The Merton (1974) model, the Leland (1994) model, and the Leland and Toft (1996) model.

The Merton model is not a model that can be appropriately applied to real bond pricing data, because it makes the assumptions that bonds do not pay coupons, and that the bond represents the entire debt of the firm. There are very few situations in which these assumptions are actually a reasonable description of the situation being modelled. In practice, bonds typically pay coupons, and equity is probably more appropriately treated as a perpetuity. Also the structure of aggregate debt is typically complicated, which makes fully modelling the cross-dependency of all claims very complicated, as e.g. pointed out by Jarrow and Turnbull (1995). Nevertheless, this model represents somewhat of a benchmark. It was implemented here following the approach of Eom, Helwege, and Huang (2004), who treat coupon bonds as a portfolio of zero-coupon Merton bonds (this is of course incorrect, because it ignores the dependence of defaults between the bonds in the portfolio).

The next theoretical model that is estimated is the model of Leland (1994). One of its defining feature is that aggregate debt and equity are modelled as perpetuities. Also, default occurs endogenously when the asset value hits the level at which shareholders are no longer willing to contribute funds to stave off financial distress. There is a question as to how to price finite maturity bonds within this framework. Here, what could be called a "quasi reduced-form approach" was chosen: Assuming that the coupon bonds being priced represent a negligible proportion of aggregate debt, we can calculate default probabilities in line with equity and aggregate debt, and use these to price the bonds in question. This has the advantage that it makes it possible to avoid compound optionality issues when pricing coupon bonds (Geske, 1977), as well as providing a reasonable approximation to a possibly very complicated debt structure that it would otherwise be impossible to model. This was first proposed by Ericsson and Reneby (2002).

The third theoretical model that is estimated is the model of Leland and Toft (1996), which builds on Leland (1994) but assumes that aggregate debt consists of finite maturity bonds paying a continuous coupon, but that only a small proportion expires every period, and is replaced by newly issued bonds, always with the same maturity. Following Eom, Helwege, and Huang (2004), any individual bond being priced is taken to be representative of aggregate debt.

From the perspective of empirical pricing, all of these models are overparameterized. It turns out that the likelihood functions are very flat with respect to many of the parameters (e.g. the level of the default boundary in a first-passage type structural model, assuming that the value of assets is also being estimated, or the drift of the asset value). It is therefore infeasible to estimate all of them. Here, most of them are fixed at plausible levels, which is the standard approach followed by the literature.

Since the likelihood function turned out to be not very sensitive to the drift of the asset value, the market price of (asset value) risk was fixed at 0.[13] In the Merton model, the time to maturity of equity was arbitrarily fixed at 20 years, as an approximation to an infinite maturity. Fixing this maturity arbitrarily is necessary since the Merton model per se does not consider the pricing of multiple bonds simultaneously, which will be necessary below. For the Leland model, the recovery to aggregate debt was set to 50%, and the corporate tax rate was assumed to be 20%. For all models, the per-bond recovery (recovery fraction of principal) was taken to be 50%.[14]

6 The Data

The data used to test the different estimation procedures and theoretical models came from several sources.

The corporate bond price data is the dataset compiled by Arthur Warga at the University of Houston from the National Association of Insurance Commissioners (NAIC). US regulations stipulate that insurance companies need to report all changes in their fixed income portfolios, including prices at which fixed income instruments where bought and sold. Insurance companies are some of the major investors in fixed income

[13] Changing the market price of risk to 0.5 had no effect on the estimates.

[14] These numbers could also be chosen according to a table such as the one given in the paper by Altman and Kishore (1996).

instruments and the data is therefore reasonably comprehensive. Also, the reported prices are actual transaction prices, and not matrix prices or quotes. The descriptive information on the bonds is obtained from the Fixed Income Securities Database (FISD) marketed by Mergent, Inc., which is a comprehensive database containing all fixed income securities issued after 1990 or appearing in NAIC-recorded bond transactions.

Over the period of 1994-2001, NAIC reports a total of 1,117,739 transactions. First, all trades with unreasonable counterparties (such as trades with the counterparty "ADJUSTMENT" etc.) are eliminated, leaving 866,434 transactions, representing about 43,330 bonds. Since often, one insurance company will buy a bond from another insurance companies, the same price can enter twice into the database, once for each side of the transactions. In order to prevent double counting, all prices for transactions for the same issue on the same date are averaged, to yield a maximum of one bond price observation per issue per date. This leaves 562,923 observations. Since the selected structural models have nothing to say about bonds with embedded optionality, sinking fund provisions and non-standard redemption agreements, all these bonds were eliminated, leaving 156,837 observations of 8,234 bonds for 1,332 issuers. Finally, government and agency bonds are eliminated, as well as bonds of financial issuers. Financial issuers are typically excluded when asset values are calculated from accounting data since their balance sheets are very different from the balance sheet of industrial issuers. In our case, it would not really be necessary to exclude them, but it is done here to facilitate comparison of the results with other studies. This leaves 88,243 observations of 3,907 bonds, for 817 issuers.

Since the point of the methodology presented here is that including additional information improves estimation, and most observations occur later in the sample, only issuers for which there were at least 50 bond price observations in 2001 were selected, leaving 50 issuers. The properties of the selected bonds and issuers are described in detail in tables 6 and 5. The issuers rating ranges from AA- (Proctor & Gamble) to B (Lucent Technologies), with BBB being the most common, and equity market capitalization ranging from around 2bn USD (Hertz), to around 300bn USD (Walmart).

For the selected issuers, the market value of equity was obtained from CRSP, and the value of total liabilities exclusive of shareholder equity (the notional value of aggregate debt) was obtained from Compustat.

Implementing corporate bond price models necessitates using risk-free interest rate data. Although most structural models assume constant interest rates (including the one utilised here), this is patently a simplifying assumption which will create problems if used to implement the model. In the implementation, a distinction was made between two types of interest rates: Those to discount individual bond payments, and those used to calculate default probabilities.

A risk-free curve (zero coupon constant maturity fitted yields) was obtained from Lehman Brothers (calculated from treasury market data). For each corporate bond, the price of a risk-free bond with the same payments was constructed artificially using the risk-free curve. The yield of this artificial risk-free bond was computed. This yield was used to discount the corporate bond in the pricing calculations. This procedure ensures that if any of the corporate bonds were (almost) risk-free, their price would equal the price of an equivalent non-defaultable bond.

There are potentially many bonds for any one particular firm, and each of these has a separate risk-free yield associated with it when these are calculated as described in the previous paragraph. The formulas for default probabilities for a given firm are predicated on a single risk-free rate. If the same default probability is desired for all bonds, a single risk-free rate for use in the formula therefore has to be used. Here, a simple arithmetic average of all the rates associated with different bonds for the same firm was chosen. Other possibilities were explored, but the results were not sensitive to the choice of this interest rate parameter.

7 Estimation

The three structural models presented above were estimated using the following types of data, assumptions on measurement errors and techniques:

- I: with equity and bond price data, allowing for measurement errors in equity, using Durbin and Koopman (1997),
- II: with equity and bond price data, not allowing for measurement errors in equity ($\sigma_E = 0$) using Chen and Scott (1993),
- III: only with equity price data, allowing for measurement errors in equity, using Durbin and Koopman (1997),
- IV: only with equity price data, and not allowing for measurement errors in equity ($\sigma_E = 0$), using Duan (1994),
- V: and only with equity price data, not allowing for measurement errors, using Ronn and Verma (1986).

The estimations were run separately for each firm, with the individual datasets starting in the beginning of 1999 and going until the date of the last bond price observation (2001-12-31 in most cases, but earlier in some cases).

To summarize the results of the estimation, I present some medians (and 1st and 3rd quartiles) of parameter estimates accross the 50 firms in Table 1. (Cross-sectional averages are reported in Table 7 in Appendix B.)

Table 1
Parameter estimates

1st quartile, median and 3rd quartile of parameter estimates in the cross section, for asset value volatility σ_η, and standard deviation of log pricing errors σ_B and σ_E, for different data sets and estimation methods. Estimation methods are I: all data (Durbin+Koopman), II: all data, $\sigma_E = 0$ (Chen+Scott), III: equity data (Durbin+Koopman), IV: equity data, $\sigma_E = 0$ (Duan), V: equity data (Ronn+Verma).

	Merton			Leland			Leland+Toft		
				σ_η (asset value volatility)					
	1st Q	*Med*	3rd Q	1st Q	*Med*	3rd Q	1st Q	*Med*	3rd Q
I	0.277	*0.332*	0.627	0.303	*0.383*	0.506	0.182	*0.220*	0.290
II	0.289	*0.345*	0.554	0.306	*0.389*	0.512	0.186	*0.220*	0.290
III	0.277	*0.319*	0.418	0.192	*0.255*	0.328	0.152	*0.203*	0.255
IV	0.288	*0.336*	0.438	0.226	*0.258*	0.335	0.155	*0.216*	0.261
V	0.286	*0.353*	0.449	0.226	*0.273*	0.324	0.144	*0.224*	0.253
				σ_B (standard deviation of log bond pricing error)					
	1st Q	*Med*	3rd Q	1st Q	*Med*	3rd Q	1st Q	*Med*	3rd Q
I	0.0587	*0.0828*	0.1204	0.0234	*0.0325*	0.0597	0.0266	*0.0369*	0.0510
II	0.0600	*0.0887*	0.1154	0.0234	*0.0325*	0.0588	0.0265	*0.0370*	0.0516
				σ_E (standard deviation of log equity pricing error)					
	1st Q	*Med*	3rd Q	1st Q	*Med*	3rd Q	1st Q	*Med*	3rd Q
I	0.0000	*0.0000*	0.0017	0.0000	*0.0000*	0.0000	0.0000	*0.0000*	0.0039
III	0.0000	*0.0000*	0.0049	0.0000	*0.0000*	0.0063	0.0000	*0.0000*	0.0056

It can be seen that including bond price information in the estimation (i.e. using methods I or II instead of II, IV or V) seems to increase the median asset value volatility (σ_η) that is estimated in the case of the Leland model (from about 0.25 to about 0.38). This reflects the fact that the model tends to underpredict spreads when estimated using only equity data. When forced to match observed spreads, asset value volatilities have to be higher. For the case of the Merton model and the Leland+Toft model, this effect does not seem to appear in the median, although looking at the 3rd quartile it is evident that including bond prices does produce high asset value volatility estimates for some firms. Table 7 shows that the average asset value volatility estimate is affected strongly in the case of the Merton model, indicating that in this case the estimates of asset value volatility are very high for some firms. Note that the asset value volatilities seem to be broadly consistent with the asset value volatilities reported by e.g. Jones, Mason, and Rosenfeld (1984).

We can test whether the estimated asset value volatility is significantly different when bond price information is used via a Hausman test. If the models are well specified, then the maximum likelihood estimator of the asset value volatility is consistent regardless of whether only equity price data is used , or whether both equity price and bond price data is used in estimation. The estimator using all data is the efficient estimator, whereas the estimator using only part of the data (the equity price data) is an inefficient estimator, such that we can compare estimators in cases I and III and in cases II and IV. Standardizing the squared difference between the inefficient and the efficient estimator by the difference in variance of the inefficient and efficient estimator produces a statistic that should be asymptotically χ^2-distributed with one degree of freedom under the joint null hypothesis that both estimators are consistent but one is efficient while the other is not.

There is a problem with the test statistic in that there is no guarantee that the estimated variance of the inefficient estimator is bigger than the estimated variance of the efficient estimator in finite samples. This is an issue especially in cases where only 50 bond prices are used; in these cases, the estimated variance of the efficient and inefficient estimate are often indistinguishable, implying that the Hausman statistic can not be meaningfully calculated. For example, for the Leland model in the case when the equity measurement error is not assumed to be zero (i.e. comparing I and III), the statistic can be calculated only for 37 out of the 50 issuers. For all of these 37 cases, however, the statistic is significant at 5%. In the case where equity measurement errors are assumed to be zero (i.e. comparing II and IV), for this and other models, the case is much the same, in that even though for a relatively large number of tests, the statistic cannot be calculated, a very large proportion of the statistics that can be calculated are significant (see Table 2), providing evidence that including bond prices does make a difference in estimates of the asset value volatility.

Table 2
Hausman test

Hausman test on the difference between asset value volatility estimates (σ_η) in the case where only equity prices are used in estimation (inefficient estimator), and both equity and bond prices are used in estimation (efficient estimator). The table reports the number of statistics (out of 50) significant at 5%. The number in brackets is the number of cases in which the statistic can be meaningfully calculated.

	Merton	Leland	Leland+Toft
Comparing I and III, $\sigma_E \neq 0$	16 *(23)*	37 *(37)*	26 *(33)*
Comparing II and IV, $\sigma_E = 0$	18 *(19)*	40 *(40)*	32 *(39)*

Also, comparing the average estimated standard deviation of measurement errors across bonds and equity, we can see that the standard deviation of bond price measurement errors is much larger. The standard deviation of the equity measurement error often is less than 1 bp, whereas the standard deviation of the bond measurement error is in the range of between 4 and 10%. This implies that apparently it is easier for structural models to match equity prices than it is to match bond prices. This could, for example, be an indication that bond spreads contain a non-default related component (Longstaff, Mithal, and Neis, 2005).

In the literature that tests structural model, the fit of the models is typically assessed using bond price derived measures, such as the absolute spread prediction error (see e.g. Jones, Mason, and Rosenfeld, 1984; Eom, Helwege, and Huang, 2004), as mentioned in the introduction. In order to see how the estimation methods compare along this dimension, the same was done here. However, in the case where bond price information was used in estimation, the last bond price observation was excluded from each firm's estimation sample. After estimation, the bond price and spread was predicted and compared to the actual bond price and spread. The actual price or spread being predicted is therefore not in the sample used to estimate. The accuracy of the (out-of-sample) predictions can then compared in the cross section of the bond price observations at the last date. The mean spread prediction error (in bp) and the root mean square error (also in bp) are summarized in Table 3. These are convenient measures of the bias, and

Table 3
Spread prediction errors (in bp)

Mean spread prediction error and root mean square spread prediction errors for the last bond price (out-of-sample) in basis points, for different data sets and estimation methods (rows) and different theoretical models (columns).

Mean error	Merton	Leland	Leland+Toft
all data (Durbin+Koopman)	−98	−50	77
all data, $\sigma_E = 0$ (Chen+Scott)	−95	−51	92
equity data, (Durbin+Koopman)	−177	−132	61
equity data, $\sigma_E = 0$ (Duan)	−175	−127	63
equity data (Ronn+Verma)	−175	−128	44
RMSE	Merton	Leland	Leland+Toft
all data (Durbin+Koopman)	156	110	165
all data, $\sigma_E = 0$ (Chen+Scott)	167	130	166
equity data, (Durbin+Koopman)	233	173	188
equity data, $\sigma_E = 0$ (Duan)	233	168	188
equity data (Ronn+Verma)	231	173	186

of the combination of the spread and bias of prediction errors.[15]

[15]For the sake of completeness, following Eom, Helwege, and Huang (2004), the means and standard deviations of the percentage pricing error (predicted bond price minus actual price over actual price), the absolute percentage pricing error, the percentage yield error (predicted yield minus actual yield over actual yield), the absolute percentage yield error, the percentage spread error (predicted spread minus actual spread over actual spread) and the absolute percentage spread error are also reported in

It can be clearly seen that including bond price data lowers the (root mean square) spread prediction errors for all models, that this effect is less marked for the Leland+Toft model, and that in terms of the RMSE, the Leland model produces the best fit (it produces a minimum of 110bp of RMSE). The Merton and Leland model underpredict spreads on average, whereas the Leland+Toft model overpredicts.[16] The Leland+Toft model also is particularly imprecise, which parallels the results of Eom, Helwege, and Huang (2004), although here, the largest outliers have an absolute size below 1000bp, whereas they have outliers as large as 4000bp. It also is the model for which adding bond price data has the smallest effect in terms of reducing errors. It seems possible that since the model is less able to generate precise predictions for bond prices, the estimation benefits less from including bond prices. Allowing for equity measurement errors lowers the error if bond prices are included for the Leland and Merton models (10-20bp), but appears to have no effect if only equity price information is used. It is also not clear that either of the two likelihood methods that is using only equity data (Durbin+Koopman and Duan) is outperforming calibration in terms of spread prediction errors.

8 Conclusion

The strength of structural models vis-à-vis reduced form models lies in the fact that they can link not only bond prices to prices of other bonds and credit derivatives, but also to prices of equity. Yet when implemented, even when the focus is on the prediction of bond spreads, typically only equity price data is used in estimation. This

the appendix (Table 8). Finally, Figure 1 illustrates the distribution of spread prediction errors with boxplots.

[16] Including bond price increases the bias in the case of the Leland+Toft model, although it decreases the RMSE.

paper illustrates how bond price data in addition to equity price data can be used to estimate structural models in a consistent manner using likelihood-based methods. It estimates three structural models using both types of data simultaneously, and shows that the models provide a closer fit to equity prices than to bond prices, but that both bond pricing errors as well as equity pricing errors are reduced when bond prices are used in estimation.

As a by-product, it is shown that allowing for equity measurement errors by itself only has a relatively small effect on estimation results.

Lastly, comparing estimated asset value volatilities for the case where only equity prices are used in estimation with the estimates in the case where both bond and equity prices are used in estimation using a Hausman test, it is shown that the estimates are typically significantly different in the two cases. This indicates that for example default probabilities calculated on the basis of model estimates could be very different depending on which data set the model is estimated on. While this casts some doubt on the validity of structural models, it also highlights the importance of not relying only on equity price data in estimating, judging and using this type of model.

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A Monte Carlo experiment

The aim of the Monte Carlo experiment is show that the method of Durbin and Koopman (1997) works and can be used for estimating structural models using both equity and bond prices simulataneously, allowing of small measurement errors in both types of prices. It is also compared to the methods of Ronn and Verma (1986) and Duan (1994) to indicate what problems these have in the given context.

Unfortunately, a Monte Carlo experiment for a numerical method like Durbin and Koopman (1997) is very computationally intensive. For a dataset of 250 observations each of equity and bond prices (corresponding to e.g. a year of daily data), the estimation of the full system takes about half an hour on a modern PC. Although not a constraint for doing a firm-by-firm estimation for 50 companies, this can hit the limits of feasibility for running a Monte Carlo experiment with, say, 10,000 artificial datasets. In view of computational constraints, only one structural model, the Leland model was used in the experiment.

A.1 Setup

Since the method of Ronn and Verma (1986) only allows for the estimation of the asset value volatility parameter and the asset value, these were the only quantities estimated by all the estimation approaches to facilitate a comparison.

To generate the Monte Carlo data, the following parameter assumptions were made: The standard deviation of the measurement errors was assumed to be 1%, the asset value volatility was set to 30% p.a. and the market price of risk was set to 0.5 (to produce a reasonable positive drift on average in the simulated asset values).

For each Monte Carlo experiment, 10,000 asset value paths of 250 periods each (to represent trading days) were simulated using the same set of parameters, all ending up at the same asset value, and corresponding prices of equity and one bond were calculated for each path (with measurement error). The estimations were run on each artificial data set corresponding to one asset value path, on all equity prices and all bond prices excluding the last bond price. This last bond price, the corresponding asset value and spread was then predicted. These were then compared to the actual bond price, spread and asset value. Also, the estimated asset value volatility was compared with the chosen volatility of 30%.

The final debt/equity ratio and the bond parameters were loosely based on the situation of K-Mart in Dec 2001: The final asset value is 12.5b $, and the face value of aggregate debt in the final period is 12b $ (the firm is highly leveraged). At the beginning of the sample, the bond has 6 years left to maturity, and pays a semiannual coupon of 5%.

As a firm comes closer to default, its bond prices become more responsive to the underlying financial situation of the firm (the asset value), whereas the price of equity becomes less responsive, as measured by the respective deltas. One would expect that including data on bond prices in the estimation would make a difference in particular for firms which are close enough to default for the default risk to be reflected in the price of the bonds. For firms which are not risky, the price of equity is likely to contain most of the relevant information, as the bonds are essentially priced as risk-free bonds.

In order to investigate the effect of this on estimation, two versions of the Monte Carlo experiment were run, with different risk-free rates (4 and 6%). Since the drift of the asset value is equal to the risk-free rate plus a risk premium (which is the same in both cases), the drift is higher in the case where the risk-free rate is set to 6%. Given that in both cases, the paths end up at the same point, on average, the starting point will be lower for the case with the greater drift. The default probability over the 250 trading days (one year) is 0.52% for the greater drift, and 0.03% for the smaller drift. Average one year default probabilities for 1920 - 2004 as reported by Moodys are 0.06% for Aa rated issuers, 0.07% for A rated issuers, 0.30% for Baa rated issuers, and 1.31% for Ba rated issuers. It can be seen that the case of $r = 6\%$ corresponds roughly to an issuer which is at the border between investment grade and non-investment grade in terms of its rating (low quality issuer), and the case of $r = 4\%$ represents the case of an issuer with a quality higher than an Aa rated issuer (high quality issuer).

Also, choosing these two cases with different drifts will highlight the difficulties that the calibration technique faces when leverage changes. Since the drift in the low quality issuer case is higher (leverage changes faster), it is possible to anticipate that the performance of the calibration technique should be worse in this case.

A.2 Results of the Monte Carlo experiment

For the Duan technique and calibration, the equity price in all periods is used to estimate the model parameter (the asset value volatility). The bond price in the last period is then predicted, based on the equity-implied asset value in the last period. For the full system estimation without restrictions via Durbin+Koopman, all equity prices as well as bond prices (except the last bond price, which we are trying to predict) are used to estimate the asset value volatility and the asset value in the last period, on which the estimate of the bond price in the last period is based. We can then compare the estimated asset value volatility, the predicted asset value, bond price and spread for the three cases. This is an out-of-sample prediction in the sense that the quantity being predicted is not in the sample used to estimate.

The results for the high quality and low quality issuer cases are presented in Table 4. It can be seen that estimating the full system using Durbin+Koopman clearly outperforms estimation using only equity prices (and assuming zero equity measurement error) with the other methods if the data is generated according to the structural model being estimated, and if there is indeed a small amount of measurement error in prices. Mean spread prediction errors are 1bp or less in size for Durbin+Koopman for the full system, and the root mean square error (RMSE) is 22 bp, whereas the mean error is in the range of 30bp for the Duan technique on equity data only (a RMSE of about 45bp). As expected, ignoring a measurement error in equity if it is present produces a bias, and not using bond price data decreases efficiency.

Table 4
Monte Carlo results

Mean error, median error and root mean square error for the different estimation methods and estimates of different quantities. The estimation methods are: Full system estimation using the method proposed by Durbin and Koopman (1997), estimation using only the simulated equity data with the method proposed by Duan (1994),+estimation using equity data as in Ronn and Verma (1986). The estimated quantities are: Asset value volatility, the asset value (in millions of dollars), the bond spread (in basis points) and the bond price (in USD per 100 USD of face value). The errors are calculated by subtracting the actual value from the estimated value. Errors are provided for a hypothetical high and low quality issuer respectively.

	High quality issuer			**Low quality issuer**		
	Mean error	Median error	RMSE	Mean error	Median error	RMSE
σ_η						
Full system	0.0002	0.0003	0.0021	0.0000	0.0000	0.0012
Duan	0.0397	0.0391	0.0777	0.0262	0.0257	0.0363
Ronn+Verma	−0.0177	−0.0166	0.0285	0.0978	0.0699	0.1765
Asset Value						
Full system	3	0	68	0	2	57
Duan	−517	−517	583	−313	−312	733
Ronn+Verma	231	239	395	−1017	−851	1519
Spread						
Full system	0	1	22	0	−1	23
Duan	36	35	76	29	27	76
Ronn+Verma	−17	−17	32	101	77	157
Bond Price						
Full system	0.00	−0.02	0.94	0.01	0.03	0.87
Duan	−1.50	−1.49	1.95	−1.08	−1.01	1.70
Ronn+Verma	0.59	0.60	1.38	−3.58	−2.84	5.49

The direction of the bias is also as expected: Since the Duan method necessarily ignores the measurement error in the equity price by construction, it will in general overestimate the asset value volatility (e.g. for independent measurement errors). This in turn means that asset values will be underestimated. Hence default probabilities are overstated, bond prices underpredicted, and the resulting spread predictions are too high.

For the parameter values chosen here, it is actually clear that the Calibration technique outperforms Duan in terms of the root mean square error (RMSE) in the case of the high quality issuer, since it has a mean spread error of -14bp, and a RMSE of 32bp. For the low quality issuer, changing leverage is more of an issue, and it is indeed clear that in this case Calibration performs much worse, with a mean error of about 100bp, and a RMSE of 157bp (Table 4).

We conclude that Durbin+Koopman estimation of the full system works as expected if the data is generated by the model, and small measurement errors are present.

B Supplementary Tables and Figures

Table 5
Summary statistics of bonds

Notional is stated in millions of USD. Time to maturity is stated in years, at the date of the prediction.

	Notional	Coupon	Time to maturity
Minimum	15	5.250	0.622
1st Quartile	150	6.563	2.528
Median	250	7.200	5.125
Mean	403	7.279	7.669
3rd Quartile	500	7.875	8.401
Maximum	1500	10.750	27.190

Table 6
Issuers

List of issuers. Rating is the S&P long term domestic issuer rating according to Compustat, Market Cap is market capitalisation on 2000-01-03 in millions of USD as given in Compustat.

Name	Rating	Market Cap	Name	Rating	Market Cap
Anadarko Pete Corp	BBB+	4,168	Lockheed Martin Corp	BBB	8,018
Anheuser Busch Cos Inc	A+	32,570	Lowes Cos Inc	A	21,314
Archer Daniels Midland Co	A+	7,300	Lucent Technologies Inc	B	242,171
AT & T Corp	BBB	170,551	Marriott Intl Inc	BBB+	7,476
Bellsouth Telecommunications Inc	A+	86,586	Masco Corp	BBB+	11,166
Boeing Co	A	37,907	May Dept Stores Co	BBB+	10,359
Charter Communications Hldgs Llc	CCC+	5,009	Motorola Inc	BBB	90,755
Coca Cola Enterprises Inc	A	8,482	Norfolk Southn Corp	BBB	7,660
COLUMBIA / Hca Healthcare Corp	BBB-	16,369	Northrop Grumman Corp	BBB	3,674
Corning Inc	BB+	29,817	Owens Ill Inc	BB-	3,546
Csx Corp	BBB	6,754	Penney J C Inc	BB+	5,127
Dayton Hudson Corp	A+	31,576	Philip Morris Cos Inc		55,456
Deere & Co	A-	10,019	Praxair Inc	A-	7,917
Delta Air Lines Inc Del	B-	6,643	Procter & Gamble Co	AA-	140,862
Disney Walt Co	BBB+	61,612	Ralston Purina Co		8,296
Dow Chem Co	A-	29,084	Raytheon Co	BBB-	6,188
Du Pont E I De Nemours & Co	AA-	67,803	Royal Caribbean Cruises Ltd	BB+	8,327
Emerson Elec Co	A	24,878	Tenet Healthcare Corp	B	7,492
Enron Corp		31,084	Union Pac Corp	BBB	10,703
Ford Mtr Co Del	BBB-	55,606	United Technologies Corp	A	29,921
Gte Corp	A+	66,385	Viacom Inc	A-	31,325
Hasbro Inc	BBB-	3,572	Wal Mart Stores Inc	AA	297,843
Hertz Corp	BBB-	2,015	Weyerhaeuser Co	BBB	16,476
Honeywell Intl Inc	A	44,739	Worldcom Inc		147,416
International Business Machs Corp	A+	208,426	Xerox Corp	BB-	16,066

Table 7
Average parameter estimates

Cross-sectional averages of estimates for asset value volatility σ_η, and standard deviation of log pricing errors σ_B and σ_E, for different data sets and estimation methods.

	Merton	Leland	Leland+Toft
σ_η (asset value volatility)			
all data (Durbin+Koopman)	0.478	0.415	0.238
all data, $\sigma_E = 0$ (Chen+Scott)	0.533	0.448	0.251
equity data (Durbin+Koopman)	0.380	0.277	0.214
equity data, $\sigma_E = 0$ (Duan)	0.399	0.291	0.222
equity data (Ronn+Verma)	0.399	0.294	0.211
σ_B (standard deviation of log bond pricing error)			
all data (Durbin+Koopman)	0.0970	0.0516	0.0471
all data, $\sigma_E = 0$ (Chen+Scott)	0.0961	0.0512	0.0472
σ_E (standard deviation of log equity pricing error)			
all data (Durbin+Koopman)	0.0019	0.0007	0.0026
equity data, (Durbin+Koopman)	0.0030	0.0034	0.0029

Table 8
Prediction errors

Means and standard deviations (numbers in brackets) of the percentage pricing error (predicted bond price minus actual price over actual price), the absolute percentage pricing error, the percentage yield error (predicted yield minus actual yield over actual yield), the absolute percentage yield error, the percentage spread error (predicted spread minus actual spread over actual spread) and the absolute percentage spread error, for the different theoretical models, and the different estimation methods, numbered as follows: 1. All data (Durbin+Koopman), 2. all data, $\sigma_E = 0$ (Chen+Scott), 3. equity data (Durbin+Koopman), 4. equity data, $\sigma_E = 0$ (Duan), 5. equity data (Ronn+Verma).

Method	% price err	abs % price err	% yield err	abs % yield err	% spread err	abs % spread err
			Merton Model			
1.	5.82	7.15	−15.35	18.73	−61.26	73.61
	(8.33)	(7.20)	(15.04)	(10.42)	(52.12)	(31.81)
2.	5.00	6.73	−18.73	23.08	−74.37	85.39
	(8.01)	(6.60)	(18.86)	(13.05)	(51.56)	(29.42)
3.	9.68	9.72	−26.13	27.29	−95.45	95.50
	(10.52)	(10.49)	(16.12)	(14.02)	(16.99)	(16.73)
4.	9.49	9.65	−25.88	27.27	−94.11	95.21
	(10.59)	(10.44)	(16.59)	(14.15)	(22.53)	(17.20)
5.	9.48	9.66	−25.86	27.28	−94.06	95.36
	(10.46)	(10.29)	(16.48)	(13.95)	(22.64)	(16.20)
			Leland Model			
1.	2.04	3.53	−9.21	12.61	−37.36	49.94
	(4.99)	(4.06)	(13.97)	(10.93)	(50.36)	(37.63)
2.	1.80	3.42	−11.65	15.86	−44.54	55.93
	(5.00)	(4.05)	(18.51)	(14.99)	(52.49)	(39.88)
3.	6.12	6.83	−20.69	22.84	−77.28	81.53
	(7.16)	(6.48)	(16.04)	(12.73)	(37.03)	(26.13)
4.	5.74	6.45	−20.15	22.31	−75.48	79.74
	(6.62)	(5.92)	(16.15)	(12.93)	(37.28)	(26.76)
5.	5.88	6.58	−20.17	22.43	−74.89	80.06
	(7.35)	(6.72)	(16.45)	(13.14)	(39.82)	(27.74)
			Leland+Toft Model			
1.	−1.29	4.21	21.49	25.68	60.28	118.43
	(5.94)	(4.36)	(53.46)	(51.54)	(235.16)	(211.45)
2.	−1.61	4.34	21.88	25.83	60.29	116.00
	(5.88)	(4.25)	(52.67)	(50.82)	(232.72)	(210.14)
3.	−0.44	5.11	19.52	26.91	53.44	118.16
	(8.50)	(6.78)	(54.13)	(50.80)	(234.23)	(208.69)
4.	−0.64	4.96	19.81	26.76	54.79	117.63
	(8.23)	(6.57)	(54.08)	(50.94)	(234.65)	(209.82)
5.	0.53	5.00	17.53	26.39	48.72	115.88
	(7.91)	(6.11)	(54.49)	(50.72)	(234.91)	(209.58)

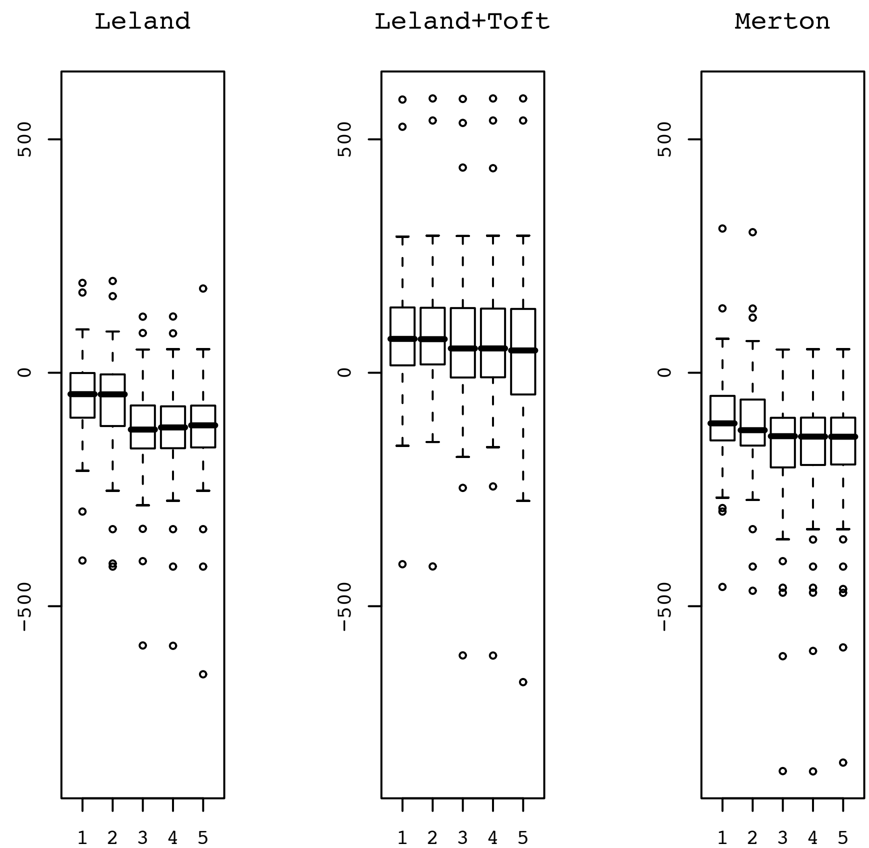

Figure 1. Boxplots of the spread prediction errors (in bp)
Boxplots of the spread prediction errors, in basis points, for different data sets and different estimation methods. The numbers indicate the following datasets and estimation methods: 1. All data (Durbin+Koopman), 2. all data, $\sigma_E = 0$ (Chen+Scott), 3. equity data (Durbin+Koopman), 4. equity data, $\sigma_E = 0$ (Duan), 5. equity data (Ronn+Verma).